Integration by Parts gives us

$$\int x^3 \ln x\,dx = (\ln x)\left(\frac{1}{4}x^4\right) - \int \left(\frac{1}{x}\right)\left(\frac{1}{4}x^4\right) dx$$
$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C = \frac{x^4}{16}(4\ln x - 1) + C.$$

6. $\int \tan^{-1} x\,dx;\quad u = \tan^{-1} x,\ v' = 1$

SOLUTION Using $u = \tan^{-1} x$ and $v' = 1$ gives us

$$u = \tan^{-1} x \qquad v = x$$
$$u' = \frac{1}{x^2+1} \qquad v' = 1$$

Integration by Parts gives us

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \left(\frac{1}{x^2+1}\right) x\,dx.$$

For the integral on the right we'll use the substitution $w = x^2 + 1$, $dw = 2x\,dx$. Then we have

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\int \left(\frac{1}{x^2+1}\right) 2x\,dx = x\tan^{-1} x - \frac{1}{2}\int \frac{dw}{w}$$
$$= x\tan^{-1} x - \frac{1}{2}\ln|w| + C = x\tan^{-1} x - \frac{1}{2}\ln|x^2+1| + C.$$

In Exercises 7–32, use Integration by Parts to evaluate the integral.

7. $\int (3x-1)e^{-x}\,dx$

SOLUTION Let $u = 3x - 1$ and $v' = e^{-x}$. Then we have

$$u = 3x - 1 \qquad v = -e^{-x}$$
$$u' = 3 \qquad v' = e^{-x}$$

Using Integration by Parts, we get

$$\int (3x-1)e^{-x}\,dx = (3x-1)(-e^{-x}) - \int (3)(-e^{-x})\,dx$$
$$= -e^{-x}(3x-1) + 3\int e^{-x}\,dx = -e^{-x}(3x-1) - 3e^{-x} + C = -e^{-x}(3x+2) + C.$$

8. $\int xe^{-x}\,dx$

SOLUTION Let $u = x$ and $v' = e^{-x}$. Then we have

$$u = x \qquad v = -e^{-x}$$
$$u' = 1 \qquad v' = e^{-x}$$

Using Integration by Parts, we get

$$\int x e^{-x}\,dx = x(-e^{-x}) - \int (1)(-e^{-x})\,dx$$
$$= -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C.$$

9. $\int x^2 e^x\,dx$

SOLUTION Let $u = x^2$ and $v' = e^x$. Then we have

$$u = x^2 \quad v = e^x$$
$$u' = 2x \quad v' = e^x$$

Using Integration by Parts, we get

$$\int x^2 e^x\,dx = x^2 e^x - 2\int xe^x\,dx.$$

We must apply Integration by Parts again to evaluate $\int xe^x\,dx$. Taking $u = x$ and $v' = e^x$, we get

$$\int xe^x\,dx = xe^x - \int (1)e^x\,dx = xe^x - e^x + C.$$

Plugging this into the original equation gives us

$$\int x^2 e^x\,dx = x^2e^x - 2\left(xe^x - e^x\right) + C = e^x(x^2 - 2x + 2) + C.$$

10. $\int x^2 \sin x\,dx$

SOLUTION Let $u = x^2$ and $v' = \sin x$. Then we have

$$u = x^2 \quad v = -\cos x$$
$$u' = 2x \quad v' = \sin x$$

Using Integration by Parts, we get

$$\int x^2 \sin x\,dx = x^2(-\cos x) - \int 2x(-\cos x)\,dx = -x^2\cos x + 2\int x\cos x\,dx.$$

We must apply Integration by Parts again to evaluate $\int x\cos x\,dx$. Taking $u = x$ and $v' = \cos x$, we get

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Plugging this into the original equation gives us

$$\int x^2 \sin x\,dx = -x^2\cos x + 2(x\sin x + \cos x) + C = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

11. $\int x\cos 2x\,dx$

SOLUTION Let $u = x$ and $v' = \cos 2x$. Then we have

$$u = x \quad v = \tfrac{1}{2}\sin 2x$$
$$u' = 1 \quad v' = \cos 2x$$

Using Integration by Parts, we get

$$\begin{aligned}\int x\cos 2x\,dx &= x\left(\frac{1}{2}\sin 2x\right) - \int (1)\left(\frac{1}{2}\sin 2x\right)dx\\ &= \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C.\end{aligned}$$

12. $\int x^2\sin(3x+1)\,dx$

SOLUTION Let $u = x^2$ and $v' = \sin(3x+1)$. Then we have

$$u = x^2 \quad v = -\tfrac{1}{3}\cos(3x+1)$$
$$u' = 2x \quad v' = \sin(3x+1)$$

Using Integration by Parts, we get

$$\int x^2 \sin(3x+1)\,dx = x^2\left(-\frac{1}{3}\cos(3x+1)\right) - \int (2x)\left(-\frac{1}{3}\cos(3x+1)\right)dx$$
$$= -\frac{1}{3}x^2\cos(3x+1) + \frac{2}{3}\int x\cos(3x+1)\,dx.$$

We must apply Integration by Parts again to evaluate $\int x\cos(3x+1)\,dx$. Using $u = x$ and $v' = \cos(3x+1)$, we get

$$\int x\cos(3x+1)\,dx = \frac{1}{3}x\sin(3x+1) - \int (1)\frac{1}{3}\sin(3x+1) = \frac{1}{3}x\sin(3x+1) + \frac{1}{9}\cos(3x+1) + C.$$

Plugging this into the original equation, we get

$$\int x^2\sin(3x+1)\,dx = -\frac{1}{3}x^2\cos(3x+1) + \frac{2}{3}\left[\frac{1}{3}x\sin(3x+1) + \frac{1}{9}\cos(3x+1)\right] + C$$
$$= -\frac{1}{3}x^2\cos(3x+1) + \frac{2}{9}x\sin(3x+1) + \frac{2}{27}\cos(3x+1) + C.$$

13. $\int e^{-x}\sin x\,dx$

SOLUTION Let $u = e^{-x}$ and $v' = \sin x$. Then we have

$$\begin{aligned} u &= e^{-x} & v &= -\cos x \\ u' &= -e^{-x} & v' &= \sin x \end{aligned}$$

Using Integration by Parts, we get

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - \int (-e^{-x})(-\cos x)\,dx = -e^{-x}\cos x - \int e^{-x}\cos x\,dx.$$

We must apply Integration by Parts again to evaluate $\int e^{-x}\cos x\,dx$. Using $u = e^{-x}$ and $v' = \cos x$, we get

$$\int e^{-x}\cos x\,dx = e^{-x}\sin x - \int (-e^{-x})(\sin x)\,dx = e^{-x}\sin x + \int e^{-x}\sin x\,dx.$$

Plugging this into the original equation, we get

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - \left[e^{-x}\sin x + \int e^{-x}\sin x\,dx\right].$$

Solving this equation for $\int e^{-x}\sin x\,dx$ gives us

$$\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + C.$$

14. $\int e^{4x}\cos 3x\,dx$

SOLUTION Let $u = e^{4x}$ and $v' = \cos 3x$. Then we have

$$\begin{aligned} u &= e^{4x} & v &= \tfrac{1}{3}\sin 3x \\ u' &= 4e^{4x} & v' &= \cos 3x \end{aligned}$$

Using Integration by Parts, we get

$$\int e^{4x}\cos 3x\,dx = e^{4x}\left(\frac{1}{3}\sin 3x\right) - \int (4e^{4x})\left(\frac{1}{3}\sin 3x\right)dx = \frac{1}{3}e^{4x}\sin 3x - \frac{4}{3}\int e^{4x}\sin 3x\,dx.$$

We must apply Integration by Parts again to evaluate $\int e^{4x}\sin 3x\,dx$. Using $u = e^{4x}$ and $v' = \sin 3x$, we get

$$\int e^{4x}\sin 3x\,dx = -\frac{1}{3}e^{4x}\cos 3x - \int 4e^{4x}\left(-\frac{1}{3}\cos 3x\right)dx = -\frac{1}{3}e^{4x}\cos 3x + \frac{4}{3}\int e^{4x}\cos 3x\,dx.$$

Plugging this into the original equation, we get

$$\int e^{4x}\cos 3x\,dx = \frac{1}{3}e^{4x}\sin 3x - \frac{4}{3}\left[-\frac{1}{3}e^{4x}\cos 3x + \frac{4}{3}\int e^{4x}\cos 3x\,dx\right]$$
$$= \frac{1}{3}e^{4x}\sin 3x + \frac{4}{9}e^{4x}\cos 3x - \frac{16}{9}\int e^{4x}\cos 3x\,dx.$$

Solving this equation for $\int e^{4x}\cos 3x\,dx$, we get

$$\int e^{4x}\cos 3x\,dx = \frac{1}{25}e^{4x}(3\sin 3x + 4\cos 3x) + C.$$

15. $\int x\ln x\,dx$

SOLUTION Let $u = \ln x$ and $v' = x$. Then we have

$$u = \ln x \qquad v = \tfrac{1}{2}x^2$$
$$u' = \tfrac{1}{x} \qquad v' = x$$

Using Integration by Parts, we get

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int\left(\frac{1}{x}\right)\left(\frac{1}{2}x^2\right)dx$$
$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C = \frac{1}{4}x^2(2\ln x - 1) + C.$$

16. $\int \frac{\ln x}{x^2}\,dx$

SOLUTION Let $u = \ln x$ and $v' = x^{-2}$. Then we have

$$u = \ln x \qquad v = -x^{-1}$$
$$u' = \tfrac{1}{x} \qquad v' = x^{-2}$$

Using Integration by Parts, we get

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x - \int \frac{1}{x}\left(\frac{-1}{x}\right)dx = -\frac{1}{x}\ln x + \int x^{-2}\,dx$$
$$= -\frac{1}{x}\ln x - \frac{1}{x} + C = -\frac{1}{x}(\ln x + 1) + C.$$

17. $\int x^{-9}\ln x\,dx$

SOLUTION Let $u = \ln x$ and $v' = x^{-9}$. Then we have

$$u = \ln x \qquad v = -\tfrac{1}{8}x^{-8}$$
$$u' = \tfrac{1}{x} \qquad v' = x^{-9}$$

Using Integration by Parts, we get

$$\int x^{-9}\ln x\,dx = -\frac{1}{8}x^{-8}\ln x - \int \frac{1}{x}\left(-\frac{1}{8}x^{-8}\right)dx = -\frac{\ln x}{8x^8} + \frac{1}{8}\int x^{-9}\,dx$$
$$= -\frac{\ln x}{8x^8} + \frac{1}{8}\left(\frac{x^{-8}}{-8}\right) + C = -\frac{1}{8x^8}\left(\ln x + \frac{1}{8}\right) + C.$$

18. $\int e^x\sin(2x)\,dx$

SOLUTION Let $u = \sin 2x$ and $v' = e^x$. Then we have

$$u = \sin 2x \qquad v = e^x$$
$$u' = 2\cos 2x \qquad v' = e^x$$

Using Integration by Parts, we get

$$\int e^x \sin 2x\, dx = e^x \sin 2x - 2\int e^x \cos 2x\, dx.$$

We must apply Integration by Parts again to evaluate $\int e^x \cos 2x\, dx$. Using $u = \cos 2x$ and $v' = e^x$, we get

$$\int e^x \cos 2x\, dx = e^x \cos 2x - \int (-2\sin 2x)e^x\, dx = e^x \cos 2x + 2\int e^x \sin 2x\, dx.$$

Plugging this into the original equation, we get

$$\int e^x \sin 2x\, dx = e^x \sin 2x - 2\left[e^x \cos 2x + 2\int e^x \sin 2x\, dx\right] = e^x \sin 2x - 2e^x \cos 2x - 4\int e^x \sin 2x\, dx.$$

Solving this equation for $\int e^x \sin 2x\, dx$ gives us

$$\int e^x \sin 2x\, dx = \frac{1}{5}e^x(\sin 2x - 2\cos 2x) + C.$$

19. $\int x\cos(2-x)\, dx$

SOLUTION Let $u = x$ and $v' = \cos(2-x)$. Then we have

$$u = x \qquad v = -\sin(2-x)$$
$$u' = 1 \qquad v' = \cos(2-x)$$

Using Integration by Parts, we get

$$\int x\cos(2-x)\, dx = -x\sin(2-x) - \int (1)(-\sin(2-x))\, dx$$
$$= -x\sin(2-x) + \int \sin(2-x)\, dx = -x\sin(2-x) + \cos(2-x) + C.$$

20. $\int x^2 \ln x\, dx$

SOLUTION Let $u = \ln x$ and $v' = x^2$. Then we have

$$u = \ln x \qquad v = \tfrac{1}{3}x^3$$
$$u' = \tfrac{1}{x} \qquad v' = x^2$$

Using Integration by Parts, we get

$$\int x^2 \ln x\, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{x}\left(\frac{1}{3}x^3\right) dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\, dx$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{x^3}{3}\right) + C = \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C.$$

21. $\int x\, 2^x\, dx$

SOLUTION Let $u = x$ and $v' = 2^x$. Then we have

$$u = x \qquad v = \frac{2^x}{\ln 2}$$

$$u' = 1 \qquad v' = 2^x$$

Using Integration by Parts, we get

$$\int x\,2^x\,dx = x\left(\frac{2^x}{\ln 2}\right) - \int (1)\frac{2^x}{\ln 2}\,dx = \frac{x\,2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x\,dx = \frac{x\,2^x}{\ln 2} - \frac{1}{\ln 2}\left(\frac{2^x}{\ln 2}\right) + C = \frac{2^x}{\ln 2}\left(x - \frac{1}{\ln 2}\right) + C.$$

22. $\int x \sec^2 x\,dx$

SOLUTION Let $u = x$ and $v' = \sec^2 x$. Then we have

$$u = x \qquad v = \tan x$$
$$u' = 1 \qquad v' = \sec^2 x$$

Using Integration by Parts, we get

$$\int x\sec^2 x\,dx = x\tan x - \int (1)\tan x\,dx = x\tan x - \ln|\sec x| + C.$$

23. $\int (\ln x)^2\,dx$

SOLUTION Let $u = (\ln x)^2$ and $v' = 1$. Then we have

$$u = (\ln x)^2 \qquad v = x$$
$$u' = \frac{2}{x}\ln x \qquad v' = 1$$

Using Integration by Parts, we get

$$\int (\ln x)^2\,dx = (\ln x)^2(x) - \int \left(\frac{2}{x}\ln x\right)x\,dx = x(\ln x)^2 - 2\int \ln x\,dx.$$

We must apply Integration by Parts again to evaluate $\int \ln x\,dx$. Using $u = \ln x$ and $v' = 1$, we have

$$\int \ln x\,dx = x\ln x - \int \frac{1}{x}\cdot x\,dx = x\ln x - \int dx = x\ln x - x + C.$$

Plugging this into the original equation, we get

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\,(x\ln x - x) + C = x\left[(\ln x)^2 - 2\ln x + 2\right] + C.$$

24. $\int \cos^{-1} x\,dx$

SOLUTION Let $u = \cos^{-1} x$ and $v' = 1$. Then we have

$$u = \cos^{-1} x \qquad v = x$$
$$u' = \frac{-1}{\sqrt{1-x^2}} \qquad v' = 1$$

Using Integration by Parts, we get

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}}\,dx.$$

We can evaluate $\int \frac{-x}{\sqrt{1-x^2}}\,dx$ by making the substitution $w = 1 - x^2$. Then $dw = -2x\,dx$, and we have

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} = x\cos^{-1} x - \frac{1}{2}\int w^{-1/2}\,dw$$
$$= x\cos^{-1} x - \frac{1}{2}(2w^{1/2}) + C = x\cos^{-1} x - \sqrt{1-x^2} + C.$$

25. $\int \sin^{-1} x\,dx$

SOLUTION Let $u = \sin^{-1} x$ and $v' = 1$. Then we have

$$u = \sin^{-1} x \qquad v = x$$
$$u' = \frac{1}{\sqrt{1-x^2}} \qquad v' = 1$$

Using Integration by Parts, we get

$$\int \sin^{-1} x\, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx.$$

We can evaluate $\int \frac{x}{\sqrt{1-x^2}}\, dx$ by making the substitution $w = 1 - x^2$. Then $dw = -2x\, dx$, and we have

$$\int \sin^{-1} x\, dx = x \sin^{-1} x + \frac{1}{2} \int \frac{-2x\, dx}{\sqrt{1-x^2}} = x \sin^{-1} x + \frac{1}{2} \int w^{-1/2}\, dw$$
$$= x \sin^{-1} x + \frac{1}{2}(2w^{1/2}) + C = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

26. $\int \sec^{-1} x\, dx$

SOLUTION We are forced to choose $u = \sec^{-1} x$, $v' = 1$, so that $u' = \frac{1}{x\sqrt{x^2-1}}$ and $v = x$. Using Integration by parts, we get:

$$\int \sec^{-1} x\, dx = x \sec^{-1} x - \int \frac{x\, dx}{x\sqrt{x^2-1}} = x \sec^{-1} x - \int \frac{dx}{\sqrt{x^2-1}}.$$

Via the substitution $\sqrt{x^2-1} = \tan\theta$ (so that $x = \sec\theta$ and $dx = \sec\theta \tan\theta d\theta$), we get:

$$\int \sec^{-1} x\, dx = x \sec^{-1} x - \int \frac{\sec\theta \tan\theta d\theta}{\tan\theta} = x \sec^{-1} x - \int \sec\theta d\theta$$
$$= x \sec^{-1} x - \ln|\sec\theta + \tan\theta| + C = x \sec^{-1} x - \ln|x + \sqrt{x^2-1}| + C.$$

27. $\int x\, 5^x\, dx$

SOLUTION Let $u = x$ and $v' = 5^x$. Then we have

$$u = x \qquad v = \frac{5^x}{\ln 5}$$
$$u' = 1 \qquad v' = 5^x$$

Using Integration by Parts, we get

$$\int x\, 5^x\, dx = x\left(\frac{5^x}{\ln 5}\right) - \int (1)\frac{5^x}{\ln 5}\, dx = \frac{x\, 5^x}{\ln 5} - \frac{1}{\ln 5}\int 5^x\, dx$$
$$= \frac{x\, 5^x}{\ln 5} - \frac{1}{\ln 5}\left(\frac{5^x}{\ln 5}\right) + C = \frac{5^x}{\ln 5}\left(x - \frac{1}{\ln 5}\right) + C.$$

28. $\int (\sin x) 5^x\, dx$

SOLUTION Let $u = 5^x$ and $v' = \sin x$. Then we have

$$u = 5^x \qquad v = -\cos x$$
$$u' = (\ln 5)5^x \qquad v' = \sin x$$

Using Integration by Parts, we get

$$\int \sin x\, 5^x\, dx = -5^x \cos x - \int (-\cos x)(\ln 5)5^x = -5^x \cos x + \ln 5 \int \cos x\, 5^x\, dx.$$

We must apply Integration by Parts again to evaluate $\int \cos x\, 5^x\, dx$. Using $u = 5^x$ and $v' = \cos x$, we get

$$\int \cos x\, 5^x\, dx = 5^x \sin x - \int (\ln 5)5^x \sin x\, dx.$$

Plugging this into the original equation, we get

$$\int \sin x\, 5^x\,dx = -5^x \cos x + \ln 5\left(5^x \sin x - \int (\ln 5)5^x \sin x\,dx\right).$$

Solving this equation for $\int \sin x\, 5^x\,dx$, we get

$$\int \sin x\, 5^x\,dx = \frac{5^x}{1+(\ln 5)^2}[(\ln 5)\sin x - \cos x] + C.$$

29. $\int x \cosh 2x\,dx$

SOLUTION Let $u = x$ and $v' = \cosh 2x$. Then we have

$$\begin{aligned} u &= x & v &= \tfrac{1}{2}\sinh 2x \\ u' &= 1 & v' &= \cosh 2x \end{aligned}$$

Using Integration by Parts, we get

$$\begin{aligned} \int x \cosh 2x\,dx &= x\left(\frac{1}{2}\sinh 2x\right) - \int (1)\left(\frac{1}{2}\sinh 2x\right)dx \\ &= \frac{1}{2}x\sinh 2x - \frac{1}{2}\int \sinh 2x\,dx = \frac{1}{2}x\sinh 2x - \frac{1}{4}\cosh 2x + C. \end{aligned}$$

30. $\int \tanh^{-1}(4x)\,dx$

SOLUTION Using $u = \tanh^{-1} 4x$ and $v' = 1$ gives us

$$\begin{aligned} u &= \tanh^{-1} 4x & v &= x \\ u' &= \frac{4}{1-16x^2} & v' &= 1 \end{aligned}$$

Integration by Parts gives us

$$\int \tanh^{-1} 4x\,dx = x\tanh^{-1} 4x - \int \left(\frac{4}{1-16x^2}\right)x\,dx.$$

For the integral on the right we'll use the substitution $w = 1-16x^2$, $dw = -32x\,dx$. Then we have

$$\begin{aligned} \int \tanh^{-1} 4x\,dx &= x\tanh^{-1} 4x + \frac{1}{8}\int \frac{dw}{w} = x\tanh^{-1} 4x + \frac{1}{8}\ln|w| + C \\ &= x\tanh^{-1} 4x + \frac{1}{8}\ln|1-16x^2| + C. \end{aligned}$$

31. $\int \sinh^{-1} x\,dx$

SOLUTION Using $u = \sinh^{-1} x$ and $v' = 1$ gives us

$$\begin{aligned} u &= \sinh^{-1} x & v &= x \\ u' &= \frac{1}{\sqrt{1+x^2}} & v' &= 1 \end{aligned}$$

Integration by Parts gives us

$$\int \sinh^{-1} x\,dx = x\sinh^{-1} x - \int \left(\frac{1}{\sqrt{1+x^2}}\right)x\,dx.$$

For the integral on the right we'll use the substitution $w = 1+x^2$, $dw = 2x\,dx$. Then we have

$$\begin{aligned} \int \sinh^{-1} x\,dx &= x\sinh^{-1} x - \frac{1}{2}\int \frac{dw}{\sqrt{w}} = x\sinh^{-1} x - \sqrt{w} + C \\ &= x\sinh^{-1} x - \sqrt{1+x^2} + C. \end{aligned}$$

32. $\int (\cos x)(\cosh x)\,dx$

SOLUTION Let $u = \cos x$ and $v' = \cosh x$. Then

$$u = \cos x \qquad v = \sinh x$$
$$u' = -\sin x \qquad v' = \cosh x$$

Integration by Parts gives us

$$\int \cos x \cosh x\,dx = \cos x \sinh x - \int (-\sin x)\sinh x\,dx = \cos x \sinh x + \int \sin x \sinh x\,dx.$$

We must apply Integration by Parts again to evaluate $\int \sin x \sinh x\,dx$. Using $u = \sin x$ and $v' = \sinh x$, we find

$$\int \sin x \sinh x\,dx = \sin x \cosh x - \int \cos x \cosh x\,dx.$$

Plugging this into the original equation, we have

$$\int \cos x \cosh x\,dx = \cos x \sinh x + \sin x \cosh x - \int \cos x \cosh x\,dx.$$

Solving this equation for $\int \cos x \cosh x\,dx$ yields

$$\int \cos x \cosh x\,dx = \frac{1}{2}(\cos x \sinh x + \sin x \cosh x) + C.$$

33. Use the substitution $u = x^{1/2}$ and then Integration by Parts to evaluate $\int e^{\sqrt{x}}\,dx$.

SOLUTION Let $w = x^{1/2}$. Then $dw = \frac{1}{2}x^{-1/2}dx$, or $dx = 2x^{1/2}\,dw = 2w\,dw$. Now,

$$\int e^{\sqrt{x}}\,dx = 2\int we^{w}\,dw.$$

Using Integration by Parts with $u = w$ and $v' = e^{w}$, we get

$$2\int we^{w}\,dw = 2(we^{w} - e^{w}) + C.$$

Substituting back, we find

$$\int e^{\sqrt{x}}\,dx = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C.$$

34. Use substitution and then Integration by Parts to evaluate $\int x^3 e^{x^2}\,dx$.

SOLUTION Let $w = x^2$. Then $dw = 2x\,dx$, and

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int we^{w}dw.$$

Using Integration by Parts, we let $u = w$ and $v' = e^{w}$. Then we have

$$\int we^{w}\,dw = we^{w} - \int (1)e^{w}\,dw = we^{w} - e^{w} + C.$$

Substituting back in terms of x, we get

$$\int x^3\,e^{x^2}\,dx = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C.$$

In Exercises 35–44, evaluate using Integration by Parts, substitution, or both if necessary.

35. $\int x\cos 4x\,dx$

SOLUTION Let $u = x$ and $v' = \cos 4x$. Then we have

$$u = x \qquad v = \tfrac{1}{4}\sin 4x$$
$$u' = 1 \qquad v' = \cos 4x$$

Using Integration by Parts, we get

$$\int x\cos 4x\,dx = \frac{1}{4}x\sin 4x - \int (1)\frac{1}{4}\sin 4x\,dx = \frac{1}{4}x\sin 4x - \frac{1}{4}\left(-\frac{1}{4}\cos 4x\right) + C$$
$$= \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C.$$

36. $\displaystyle\int \frac{\ln(\ln x)\,dx}{x}$

SOLUTION Let $w = \ln x$. Then $dw = dx/x$, and we have

$$\int \frac{\ln(\ln x)\,dx}{x} = \int \ln w\,dw$$

Now we can use Integration by Parts, letting $u = \ln w$ and $v' = 1$. Then $u' = 1/w$, $v = w$, and

$$\int \ln w\,dw = w\ln w - \int \frac{1}{w}(w)\,dw = w\ln w - w + C.$$

Substituting back in terms of x, we get

$$\int \frac{\ln(\ln x)\,dx}{x} = (\ln x)\ln(\ln x) - \ln x + C.$$

37. $\displaystyle\int \frac{x\,dx}{\sqrt{x+1}}$

SOLUTION Let $u = x + 1$. Then $du = dx$, $x = u - 1$, and

$$\int \frac{x\,dx}{\sqrt{x+1}} = \int \frac{(u-1)\,du}{\sqrt{u}} = \int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right)du = \int (u^{1/2} - u^{-1/2})\,du$$
$$= \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C.$$

38. $\displaystyle\int \sin(\ln x)\,dx$

SOLUTION Let $u = \sin(\ln x)$ and $v' = 1$. Then we have

$$u = \sin(\ln x) \qquad v = x$$
$$u' = \frac{\cos(\ln x)}{x} \qquad v' = 1$$

Using Integration by Parts, we get

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int (x)\frac{\cos(\ln x)}{x}\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

We must use Integration by Parts again to evaluate $\int \cos(\ln x)\,dx$. Let $u = \cos(\ln x)$ and $v' = 1$. Then

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \left[x\cos(\ln x) - \int(-\sin(\ln x))\,dx\right]$$
$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx.$$

Solving this equation for $\int \sin(\ln x)\,dx$, we get

$$\int \sin(\ln x)\,dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$$

39. $\displaystyle\int \cos x\ln(\sin x)\,dx$

SOLUTION Let $w = \sin x$. Then $dw = \cos x\,dx$, and

$$\int \cos x\ \ln(\sin x)\,dx = \int \ln w\,dw.$$

Now use Integration by Parts with $u = \ln w$ and $v' = 1$. Then $u' = 1/w$ and $v = w$, which gives us

$$\int \cos x\ \ln(\sin x)\,dx = \int \ln w\,dw = w\ln w - w + C = \sin x\ln(\sin x) - \sin x + C.$$

40. $\int x^3(x^2+1)^{12}\,dx$

SOLUTION Let $u = x^2 + 1$. Then $du = 2x\,dx$, $x^2 = u - 1$, and

$$\int x^3(x^2+1)^{12}\,dx = \frac{1}{2}\int x^2(x^2+1)^{12}2x\,dx = \frac{1}{2}\int (u-1)u^{12}\,du = \frac{1}{2}\left(\int u^{13}\,du - \int u^{12}\,du\right)$$
$$= \frac{1}{2}\left(\frac{1}{14}u^{14} - \frac{1}{13}u^{13}\right) + C = \frac{1}{28}(x^2+1)^{14} - \frac{1}{26}(x^2+1)^{13} + C.$$

41. $\int \sin\sqrt{x}\,dx$

SOLUTION First use substitution, with $w = \sqrt{x}$ and $dw = dx/(2\sqrt{x})$. This gives us

$$\int \sin\sqrt{x}\,dx = \int \frac{(2\sqrt{x})\sin\sqrt{x}\,dx}{(2\sqrt{x})} = 2\int w\sin w\,dw.$$

Now use Integration by Parts, with $u = w$ and $v' = \sin w$. Then we have

$$\int \sin\sqrt{x}\,dx = 2\int w\sin w\,dw = 2\left(-w\cos w - \int -\cos w\,dw\right)$$
$$= 2(-w\cos w + \sin w) + C = 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + C.$$

42. $\int \sqrt{x}e^{\sqrt{x}}dx$

SOLUTION Let $w = \sqrt{x}$. Then $dw = \frac{1}{2\sqrt{x}}\,dx$ and

$$\int \sqrt{x}e^{\sqrt{x}}\,dx = 2\int w^2e^w\,dw.$$

Now, use Integration by Parts with $u = w^2$ and $v' = e^w$. This gives

$$\int \sqrt{x}e^{\sqrt{x}}\,dx = 2\int w^2e^w\,dw = 2w^2e^w - 4\int we^w\,dw.$$

We need to use Integration by Parts again, this time with $u = w$ and $v' = e^w$. We find

$$\int we^w\,dw = we^w - \int e^w\,dw = we^w - e^w + C;$$

finally,

$$\int \sqrt{x}e^{\sqrt{x}}\,dx = 2w^2e^w - 4we^w + 4e^w + C = 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C.$$

43. $\int \frac{\ln(\ln x)\ln x\,dx}{x}$

SOLUTION Let $w = \ln x$. Then $dw = dx/x$, and

$$\int \frac{\ln(\ln x)\ln x\,dx}{x} = \int w\ln w\,dw.$$

Now use Integration by Parts, with $u = \ln w$ and $v' = w$. Then,

$$u = \ln w \qquad v = \frac{1}{2}w^2$$

$$u' = w^{-1} \quad v' = w$$

and

$$\int \frac{\ln(\ln x)\ln x\,dx}{x} = \frac{1}{2}w^2 \ln w - \frac{1}{2}\int w\,dw = \frac{1}{2}w^2 \ln w - \frac{1}{2}\left(\frac{w^2}{2}\right) + C$$
$$= \frac{1}{2}(\ln x)^2 \ln(\ln x) - \frac{1}{4}(\ln x)^2 + C = \frac{1}{4}(\ln x)^2[2\ln(\ln x) - 1] + C.$$

44. $\int x \tan^{-1} x\,dx$

SOLUTION Using Integration by Parts with $u = \tan^{-1} x$ and $v' = x$, we find

$$\int x\tan^{-1} x\,dx = \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$
$$= \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C.$$

In Exercises 45–50, compute the definite integral.

45. $\int_0^2 xe^{9x}\,dx$

SOLUTION Let $u = x$ and $v' = e^{9x}$. Then $u' = 1$ and $v = \frac{1}{9}e^{9x}$. Using Integration by Parts,

$$\int_0^2 xe^9x\,dx = \frac{1}{9}xe^{9x}\Big|_0^2 - \int_0^2 (1)\frac{1}{9}e^{9x}\,dx = \left(\frac{1}{9}xe^{9x} - \frac{1}{81}e^{9x}\right)\Big|_0^2$$
$$= \left(\frac{2}{9}e^{18} - \frac{1}{81}e^{18}\right) - \left(0 - \frac{1}{81}(1)\right) = \frac{1}{81}(18e^{18} - e^{18} + 1) = \frac{1}{18}(17e^{18} + 1).$$

46. $\int_1^3 \ln x\,dx$

SOLUTION Let $u = \ln x$ and $v' = 1$. Then $u' = 1/x$ and $v = x$. Using Integration by Parts,

$$\int_1^3 \ln x\,dx = (x\ln x - x)\Big|_1^3 = 3\ln 3 - 3 - ((1)\ln 1 - 1) = 3\ln 3 - 2.$$

47. $\int_0^4 x\sqrt{4-x}\,dx$

SOLUTION Let $u = 4 - x$. Then $x = 4 - u$, $du = -dx$ and

$$\int_0^4 x\sqrt{4-x}\,dx = \int_{u=4}^{u=0} (4-u)u^{1/2}\,(-du) = -\int_{u=4}^{u=0} (4u^{1/2} - u^{3/2})\,du$$
$$= \int_{u=0}^{u=4} (4u^{1/2} - u^{3/2})\,du = \left((4)\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right)\Big|_0^4$$
$$= \frac{8}{3}(4^{3/2}) - \frac{2}{5}(4^{5/2}) = \frac{64}{3} - \frac{64}{5} = \frac{128}{15}.$$

48. $\int_0^{\pi/4} x\sin(2x)\,dx$

SOLUTION Let $u = x$ and $v' = \sin 2x$. Then $u' = 1$ and $v = -\frac{1}{2}\cos 2x$. Using Integration by Parts,

$$\int_0^{\pi/4} x\sin(2x)\,dx = -\frac{1}{2}x\cos 2x\Big|_0^{\pi/4} - \int_0^{\pi/4}\left(-\frac{1}{2}\cos 2x\right)dx = \left(-\frac{1}{2}x\cos 2x + \left(\frac{1}{2}\right)\frac{\sin 2x}{2}\right)\Big|_0^{\pi/4}$$
$$= \left(-\frac{1}{2}\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right) - (0+0) = \frac{1}{4}.$$

49. $\int_1^4 \sqrt{x}\ln x\,dx$

SOLUTION Let $u = \ln x$ and $v' = \sqrt{x}$. Then $u' = 1/x$ and $v = \frac{2}{3}x^{3/2}$. Using Integration by Parts,

$$\int_1^4 \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x\Big|_1^4 - \int_1^4 \frac{2}{3}x^{1/2}\,dx = \left(\frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\cdot\frac{2}{3}x^{3/2}\right)\Big|_1^4$$

$$= \frac{2}{3}x^{3/2}\left(\ln x - \frac{2}{3}\right)\Big|_1^4 = \frac{2}{3}4^{3/2}\left(\ln 4 - \frac{2}{3}\right) - \frac{2}{3}(1)\left(0 - \frac{2}{3}\right) = \frac{16}{3}\ln 4 - \frac{28}{9}.$$

50. $\int_0^1 \tan^{-1} x\,dx$

SOLUTION Let $u = \tan^{-1} x$ and $v' = 1$. Then we have

$$u = \tan^{-1} x \qquad v = x$$
$$u' = \frac{1}{x^2+1} \qquad v' = 1$$

Integration by Parts gives us

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \left(\frac{1}{x^2+1}\right) x\,dx.$$

For the integral on the right we'll use the substitution $w = x^2 + 1$, $dw = 2x\,dx$. Then we have

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\int \frac{dw}{w} = x\tan^{-1} x - \frac{1}{2}\ln|w| + C = x\tan^{-1} x - \frac{1}{2}\ln|x^2+1| + C.$$

Now we can compute the definite integral:

$$\int_0^1 \tan^{-1} x\,dx = \left(x\tan^{-1} x - \frac{1}{2}\ln|x^2+1|\right)\Big|_0^1 = \left((1)\tan^{-1}(1) - \frac{1}{2}\ln 2\right) - (0) = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

51. Use Eq. (5) to evaluate $\int x^4 e^x\,dx$.

SOLUTION

$$\int x^4e^x\,dx = x^4e^x - 4\int x^3e^x\,dx = x^4e^x - 4\left[x^3e^x - 3\int x^2e^x\,dx\right]$$

$$= x^4e^x - 4x^3e^x + 12\int x^2e^x\,dx = x^4e^x - 4x^3e^x + 12\left[x^2e^x - 2\int xe^x\,dx\right]$$

$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24\int xe^x\,dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24\left[xe^x - \int e^x\,dx\right]$$

$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24\left[xe^x - e^x\right] + C.$$

Thus,

$$\int x^4e^x\,dx = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C.$$

52. Use substitution and then Eq. (5) to evaluate $\int x^4e^{7x}\,dx$.

SOLUTION Let $u = 7x$. Then $du = 7dx$, and

$$\int x^4e^{7x}\,dx = \frac{1}{7^5}\int (7x)^4e^{7x}(7dx) = \frac{1}{7^5}\int u^4e^u\,du.$$

Now use the result from Exercise 51:

$$\int x^4e^{7x}\,dx = \frac{1}{7^5}e^u[u^4 - 4u^3 + 12u^2 - 24u + 24] + C$$

$$= \frac{1}{7^5}e^{7x}[(7x)^4 - 4(7x)^3 + 12(7x)^2 - 24(7x) + 24] + C$$

$$= \frac{1}{7^5}e^{7x}[2401x^4 - 1372x^3 + 588x^2 - 168x + 24] + C.$$

53. Find a reduction formula for $\int x^n e^{-x}\,dx$ similar to Eq. (5).

SOLUTION Let $u = x^n$ and $v' = e^{-x}$. Then

$$u = x^n \qquad v = -e^{-x}$$
$$u' = nx^{n-1} \qquad v' = e^{-x}$$

Using Integration by Parts, we get

$$\int x^n e^{-x}\,dx = -x^n e^{-x} - \int nx^{n-1}(-e^{-x})\,dx = -x^n e^{-x} + n\int x^{n-1}e^{-x}\,dx.$$

54. Evaluate $\int x^n \ln x\,dx$ for $n \neq -1$. Which method should be used to evaluate $\int x^{-1}\ln x\,dx$?

SOLUTION Let $u = \ln x$ and $v' = x^n$. Then we have

$$u = \ln x \qquad v = \frac{x^{n+1}}{n+1}$$
$$u' = \frac{1}{x} \qquad v' = x^n$$

and

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{1}{x}\cdot\frac{x^{n+1}}{n+1}\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n\,dx$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\cdot\frac{x^{n+1}}{n+1} = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C.$$

For $n = -1$, $\int x^{-1}\ln x\,dx$, use the substitution $u = \ln x$, $du = dx/x$. Then

$$\int x^{-1}\ln x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C.$$

In Exercises 55–62, indicate a good method for evaluating the integral (but do not evaluate). Your choices are algebraic manipulation, substitution (specify u and du), and Integration by Parts (specify u and v′). If it appears that the techniques you have learned thus far are not sufficient, state this.

55. $\int \sqrt{x}\ln x\,dx$

SOLUTION Use Integration by Parts, with $u = \ln x$ and $v' = \sqrt{x}$.

56. $\int \frac{x^2 - \sqrt{x}}{2x}\,dx$

SOLUTION Use algebraic manipulation:

$$\frac{x^2 - \sqrt{x}}{2x} = \frac{x}{2} - \frac{1}{2\sqrt{x}}.$$

57. $\int \frac{x^3}{\sqrt{4 - x^2}}\,dx$

SOLUTION Use substitution, followed by algebraic manipulation: Let $u = 4 - x^2$. Then $du = -2x\,dx$, $x^2 = 4 - u$, and

$$\int \frac{x^3}{\sqrt{4 - x^2}}\,dx = -\frac{1}{2}\int \frac{(x^2)(-2x\,dx)}{\sqrt{u}} = -\frac{1}{2}\int \frac{(4 - u)(du)}{\sqrt{u}} = -\frac{1}{2}\int \left(\frac{4}{\sqrt{u}} - \frac{u}{\sqrt{u}}\right)du.$$

58. $\int \frac{dx}{\sqrt{4 - x^2}}$

SOLUTION The techniques learned so far are insufficient. This one requires the technique of trigonometric substitution.

59. $\int \frac{2x + 3}{x^2 + 3x + 6}\,dx$